MATH1520 University Mathematics for Applications **Fall 2021**

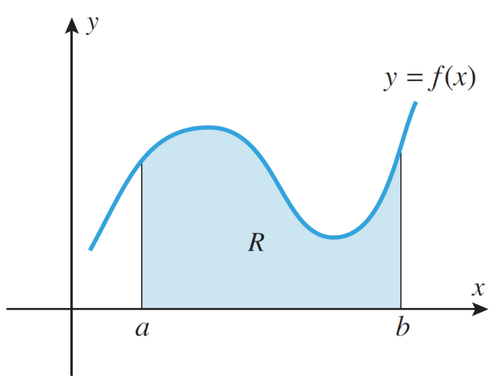
Chapter 10: Definite Integrals

Learning Objectives:
(1) Define the definite integral and explore its properties.
(2) State the fundamental theorem of calculus, and use it to compute definite integrals.
(3) Use integration by parts and by substitution to find integrals.
(4) Evaluate improper integrals with infinite limits of integration.

1 Riemann Sums & Definite Integrals

Suppose f is a function on $[a, b]$. Suppose further that $f(x)$ is positive on $[a, b]$. The we define

$$\int_a^b f(x)\,dx = \text{ area between the graph of } f(x) \text{ and the } x\text{-axis.}$$



What if some of the value of $f(x)$ is negative? Because $f(x)$ is negative, the "height" of $f(x)$ at this point is negative, so we take the area as negative. Therefore, we have the following definition.

Definition 1.1 (Total Signed Area). Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area from** $x = a$ **to** $x = b$ **under** f is: the area under the graph of f and above the x-axis on $[a, b]$ $-$ the area above the graph of f and under the x-axis on $[a, b]$.

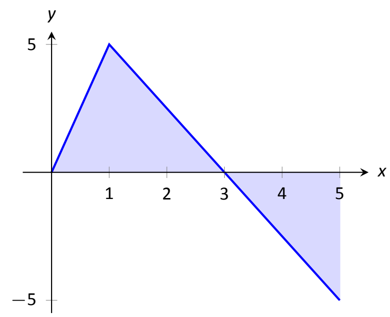
Geometric interpretation of integration The **definite integral of f on** $[a, b]$ is the total signed area under f on from a to b, denoted

$$\int_a^b f(x)\, dx,$$

where a and b are the **bounds (or limits) of integration**.

We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under the graph of f".

Example 1.1. Consider the function f given below. Compute $\int_0^5 f(x)\, dx$.



Solution. The graph of f is above the x-axis over $[0, 3]$. The area is $\frac{1}{2} \times 3 \times 5 = 7.5$.

The graph of f is under the x-axis over $[3, 5]$. This is the "negative" area. The area is $-\frac{1}{2} \times 2 \times 5 = -5$. Hence

$$\int_0^5 f(x)dx = 7.5 - 5 = 2.5.$$

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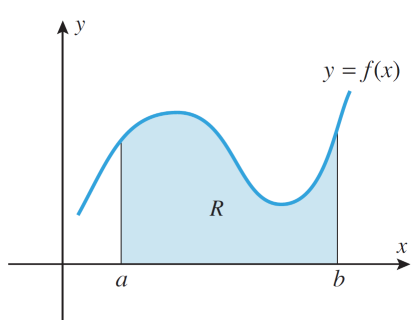
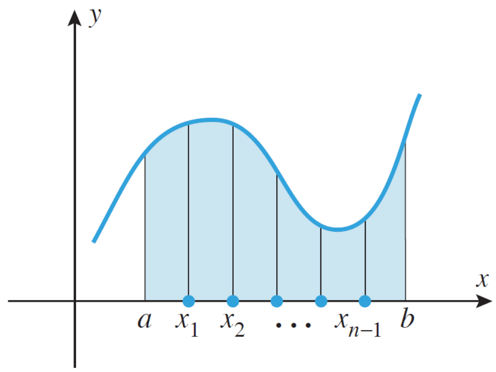
What if the region is not as simple as the previous example, such as the one below?



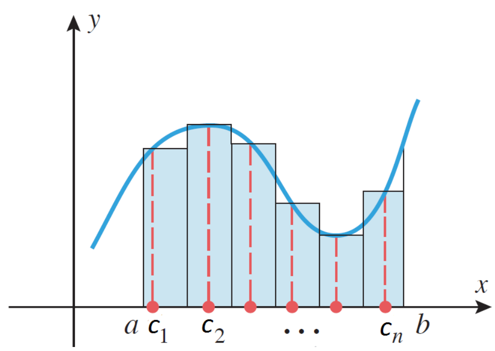
Idea: Approximate the area by small rectangles!

1. A partition of $[a, b]$: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, $x_k = \dfrac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$ divides $[a, b]$ into n subintervals $[a_{k-1}, a_k]$ with width:

$$\Delta x_k = x_k - x_{k-1} = \frac{b-a}{n}, \quad k = 1, 2, \ldots, n.$$



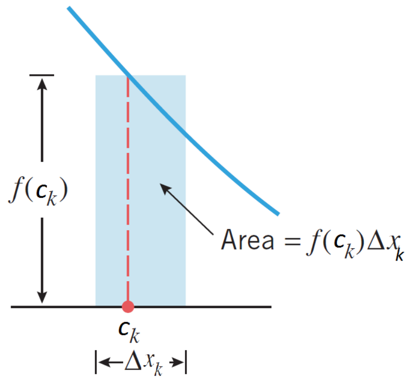
2. Choose points $c_k \in [x_{k-1}, x_k]$, $k = 1, 2, \ldots, n$, to form small rectangles.



3. Calculate the area of each rectangle and sum them up.

   For the kth subinterval,

$$\boxed{\text{Area of } k\text{th rectangle} = \text{height} \times \text{width} = f(c_k)\Delta x_k}$$



Definition 1.2.

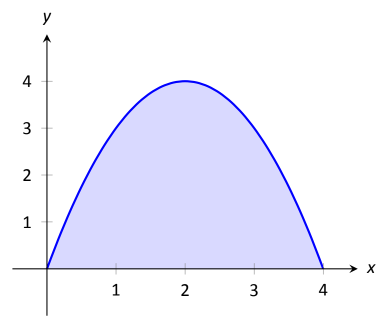
$$\boxed{\sum_{k=1}^{n} f(c_k)\Delta x_k}$$

is called a Riemann Sum of f on $[a, b]$.

In particular,

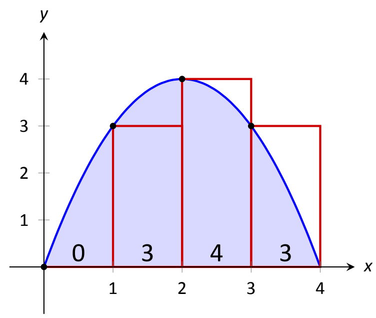
if	$c_k = x_{k-1}$, the sum is called	left Riemann sum
if	$c_k = x_k$,	right Riemann sum
if	$c_k = \dfrac{x_{k-1} + x_k}{2}$,	mid-point Riemann sum

Example 1.2. Approximate the area under $y = -x^2 + 4x$ on $[0, 4]$ with partition $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$.

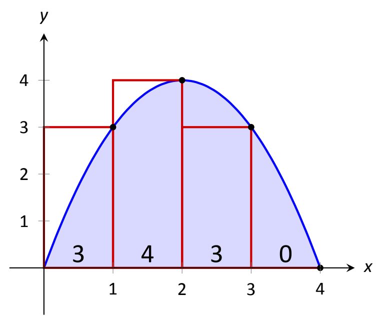


1. Left Riemann sum: $c_1 = 0, c_2 = 1, c_3 = 2, c_4 = 3$.

$$\text{Area} \approx f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 = 10.$$



2. Right Riemann sum: $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4.$



$$\text{Area} \approx f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 = 10.$$

3. Mid-point Riemann sum: $c_1 = ; c_2 = 2, c_3 = 3, c_4 = 4.$

$$\text{Area} \approx f(0.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1 = 11.$$

Question: How to get better approximation of the area?

Idea: Increase number of rectangles.

Definition 1.3. Let $f(x)$ be continuous on $[a, b]$. Consider the partition: $x_k = \frac{b-a}{n}k + a$, $k = 0, 1, \ldots, n$. For any $c_k \in [x_{k-1}, x_k], k = 1, 2, \ldots, n$, $\lim_{n\to+\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k$ is a fixed number, called definite (Riemann) integral of $f(x)$ on $[a, b]$, denoted by $\int_a^b f(x)\, dx$, i.e.,

$$\boxed{\lim_{n\to+\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)\, dx}$$

Hard Theorem: Let f be a piecewise continuous function, then $\int_a^b f(x)\,dx$ is well-defined. I.e. The limit in the preceding definition exists, and is independent of the choices of c_k.

Remark. The "Lebesque integral" is well-defined for more general functions.

Example 1.3. Evaluate $\int_2^3 x\,dx$ using the left Riemann sum with n equally spaced subintervals.

Example 1.4. Evaluate $\int_0^1 x^2\,dx$ using the right Riemann sum with n equally spaced subintervals.

Solution. Let $f(x) = x^2$. Consider the partition of $[0,1]$: $x_k = \frac{k}{n}, k = 0, \dots, n$.
Right Riemann sum: on $[x_{k-1}, x_k]$, $c_k = x_k = \dfrac{k}{n}$.

$$\sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n} = \frac{(n+1)(2n+1)}{6n^2}.$$

$(\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6})$

So, $\displaystyle\int_0^1 x^2\,dx = \lim_{n\to+\infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$ ■

Remark. It's so complicated to used definition to compute $\int_a^b f(x)\,dx$. Later, we will discuss another easier method: fundamental theorem of calculus.

Using the interpretation of definite integrals as signed areas and its definition as limits of Riemann sums, we have:

Theorem 1.1 (Properties of definite integrals)**.**

1. $\boxed{\int_a^a f(x)\,dx = 0}$

2. $\boxed{\int_a^b k\,dx = k(b-a)}$

3. $\boxed{\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx}$

   $\boxed{\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx}$

4. *if* $a < b$,

   $\boxed{\int_b^a f(x)\,dx \triangleq -\int_a^b f(x)\,dx}$ ($\triangleq$, *defined to be*)

5. $\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx}$

6. *if* $f(x) \leq g(x)$ *on* $[a, b]$, *then*

$$\boxed{\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx}$$

2 The fundamental Theorem of Calculus

Notation:

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$: definite integral of function f on $[a, b]$, which is a number.

$\int_a^x f(t)\,dt$: definite integral of function f on $[a, x]$, it can be regarded as a function of x.

Theorem 2.1 (Fundamental Theorem of Calculus)**.**

Assume $f(x)$ is continuous.

1. $\boxed{\dfrac{d}{dx}\int_a^x f(t)\,dt = f(x)}$ *(i.e. $\int_a^x f(t)\,dt$ is an anti-derivative of $f(x)$)*
2. *Let $F(x)$ be any anti-derivative of $f(x)$, $F'(x) = f(x)$, then*
   $\boxed{\int_a^b f(x)\,dx = F(x)\big|_a^b := F(b) - F(a)}$.

Heuristic explanation:

Example 2.1. Compute the integrals in Examples 1.3 and 1.4 using the fundamental theorem of calculus.

Example 2.2. Derive Theorem 1.1 from the corresponding theorem for indefinite integrals and the fundamental theorem of calculus.

Remark.

1. Differentiation $\xleftrightarrow{\text{Fundamental thm of calculus}}$ Integration

   $F'(x) = f(x)$ $\qquad$ $\int_a^b f(x)\,dx = F(b) - F(a)$

2. Anti-derivative $F(x)$ is not unique. Which one should we choose?

   Another anti-derivative: $\tilde{F}(x) = F(x) + C$, then

   $$\tilde{F}(b) - \tilde{F}(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a).$$

   so, it does not matter, we can choose any anti-derivative.

Example 2.3.

$$\int_1^9 \sqrt{x}\,dx = \int x^{1/2}\,dx\Big|_1^9 = \frac{2}{3}x^{3/2}\Big|_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}.$$

Example 2.4. Evaluate $\int_1^2 \ln x\,dx$.

We first find one antiderivative of $\ln x$,

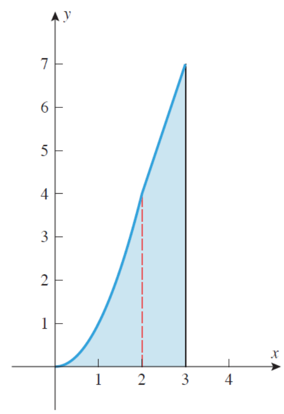
$$\begin{aligned} \int \ln x\,dx &= x\ln x - \int 1\,dx \qquad \text{(integration by parts)} \\ &= x\ln x - x + C. \end{aligned}$$

So, $\int_1^2 \ln x\,dx = (x\ln x - x)|_1^2 = 2\ln 2 - 1.$

Example 2.5. Let

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}.$$

Find $\int_0^3 f(x)\,dx$.



$$\begin{aligned}\int_0^3 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x-2)\,dx \qquad \text{(integrate separately)}\\ &= \frac{x^3}{3}\Big|_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6}.\end{aligned}$$

Exercise 2.1.

1. $\displaystyle\int_0^1 2xe^{x^2}\,dx = e - 1.$
2. $\displaystyle\int_{-1}^2 |x|\,dx = \frac{5}{2}.$

Example 2.6. Compute $\dfrac{d}{dx}$ for (1) $\displaystyle\int_1^x e^{t^2}\,dt$, (2) $\displaystyle\int_{x^2}^{x^3} e^{t^2}\,dt$, (3) $\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt$.

Solution. It's impossible to get explicit formula for $F(t) = \int e^{t^2}\,dt$.

1. By fundamental theorem of calculus (1), we have
$$\frac{d}{dx}\int_1^x e^{t^2}\,dt = e^{x^2}.$$
2. Let $F'(t) = e^{t^2}$, then
$$\frac{d}{dx}\int_{x^2}^{x^3} e^{t^2}\,dt = \frac{d}{dx}(F(x^3) - F(x^2)) = F'(x^3)\cdot 3x^2 - F'(x^2)\cdot 2x = e^{x^6}\cdot 3x^2 - e^{x^4}\cdot 2x.$$

3. Let $F'(t) = f(t)$,

$$\begin{aligned} \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt &= \frac{d}{dx}(F(h(x)) - F(g(x))) \\ &= F'(h(x))\cdot h'(x) - F'(g(x))\cdot g'(x) \\ &= f(h(x))h'(x) - f(g(x))g'(x). \end{aligned}$$

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Exercise 2.2. $\frac{d}{dx}\int_{2x}^{x+1} e^{\sqrt{t}}\,dt = e^{\sqrt{x+1}} - 2e^{\sqrt{2x}}$.

3 Definite Integration by Substitution & Integration by Parts

Theorem 3.1.

$$\boxed{\int_a^b f(g(x))g'(x)\,dx \overset{g(x)=u}{=\!=\!=} \int_{g(a)}^{g(b)} f(u)\,du}$$

Example 3.1.

1.

$$\begin{aligned} \int_0^1 8x(x^2+1)dx &= \int_0^1 4(x^2+1)\,d(x^2+1) \\ &= \int_1^2 4u\,du \quad (x^2+1=u, (0)^2+1=1, 1^2+1=2) \\ &= 2u^2\big|_1^2 \\ &= 2\times 2^2 - 2\times 1^2 = 6. \end{aligned}$$

2.

$$\begin{aligned} \int_e^{e^2} \frac{1}{x\ln x}dx &= \int_e^{e^2} \frac{1}{\ln x}\,d(\ln x) \\ &= \int_1^2 \frac{1}{u}du \quad (\ln x = u, \ln e = 1, \ln e^2 = 2) \\ &= \ln u\big|_1^2 \\ &= \ln 2 - \ln 1 = \ln 2. \end{aligned}$$

Theorem 3.2.

$$\boxed{\int_a^b u(x)\,d(v(x)) = u(x)v(x)\Big|_a^b - \int_a^b v(x)\,d(u(x))}$$

Example 3.2.

1.

$$\begin{aligned}
\int_1^e x\ln x\,dx &= \int_1^e \ln x d\left(\frac{x^2}{2}\right)\\
&= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\,d\ln x\\
&= \left(\frac{e^2}{2}\ln e - \frac{1}{2}\ln 1\right) - \int_1^e \frac{x}{2}\,dx\\
&= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e\\
&= \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right)\\
&= \frac{e^2}{4} + \frac{1}{4}.
\end{aligned}$$

2.

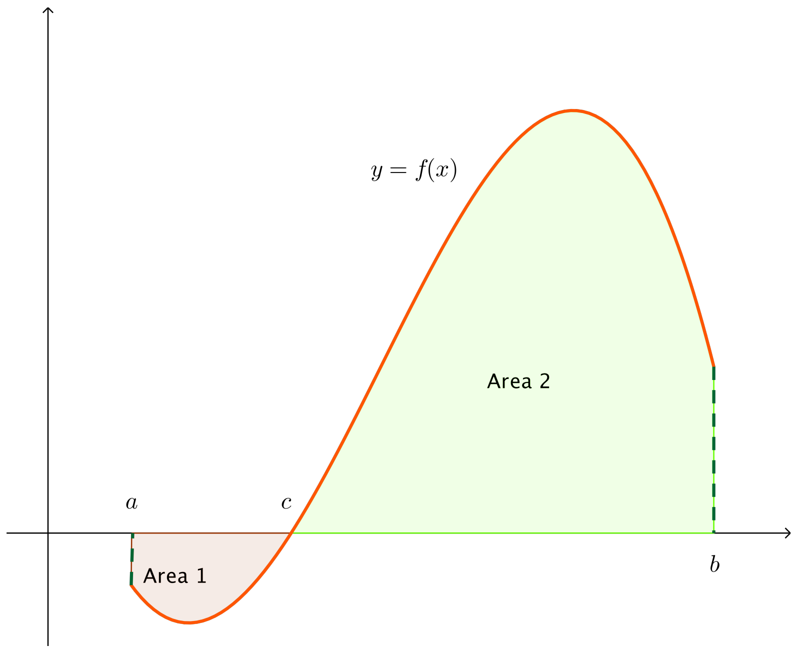
$$\begin{aligned}
\int_0^1 xe^x\,dx &= \int_0^1 x\,d(e^x)\\
&= xe^x|_0^1 - \int_0^1 e^x\,dx\\
&= e - e^x|_0^1 = 1
\end{aligned}$$

Exercise 3.1. 1. $\displaystyle\int_{-2}^1 \frac{2x+1}{x^2+x+1}dx = ?$

2. $\displaystyle\int_2^3 \frac{dx}{(x-1)(x^2+2x-3)} = ?$

4 Applications of Definite Integration

4.1 Area bounded by the graph of $f(x)$ and the x-axis on $[a, b] = \int_a^b |f(x)|\,dx$



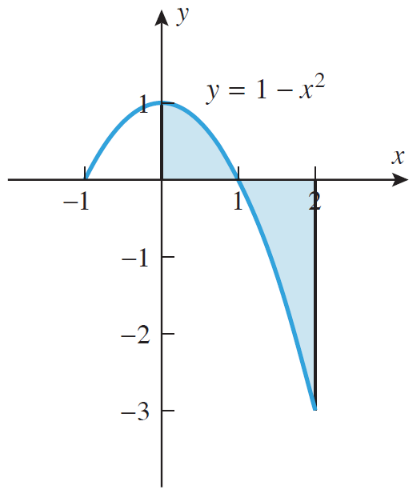
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = -\text{ Area 1} + \text{ Area 2 } = \text{Signed area}$$

$$\int_a^b |f(x)|\,dx = \int_a^c -f(x)\,dx + \int_c^b f(x)\,dx = \text{ Area 1} + \text{ Area 2 } = \text{Area}$$

Example 4.1. Find the total area between the curve $y = 1 - x^2$ and the x-axis over the interval $[0, 2]$.

Solution. Let $1 - x^2 = 0, \quad \Rightarrow \quad x = \pm 1.$

$$1 - x^2 \begin{cases} \geq 0, & \text{for } -1 \leq x \leq 1, \\ < 0, & \text{for } x < -1 \quad \text{or} \quad x > 1. \end{cases}$$



The area is given by

$$\begin{aligned}\int_0^2 |1-x^2|\,dx &= \int_0^1 (1-x^2)dx + \int_1^2 -(1-x^2)dx \\ &= \left[x-\frac{x^3}{3}\right]_0^1 - \left[x-\frac{x^3}{3}\right]_1^2 \\ &= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2.\end{aligned}$$

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Exercise 4.1. Area bounded by the graph of $f(x) = x - \sqrt{x}$ and x-axis on $[0, 2]$.

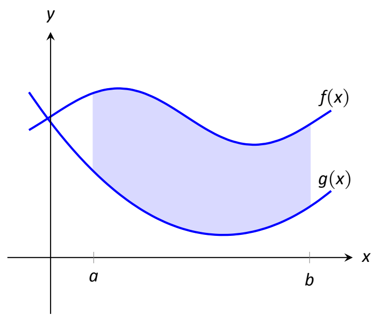
4.2 Area bounded by the graphs of $f(x)$, $g(x)$ on $[a, b] = \int_a^b |f(x) - g(x)|\,dx$

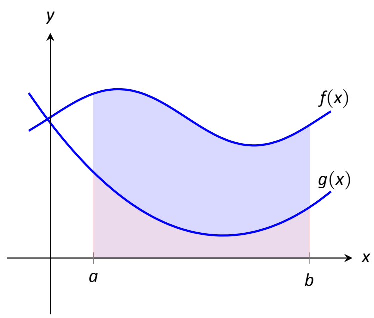
Theorem 4.1. *Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is*

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Proof. The area between $f(x)$ and $g(x)$ is obtained by subtracting the area under g from the area under f. Thus the area is

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))dx.$$





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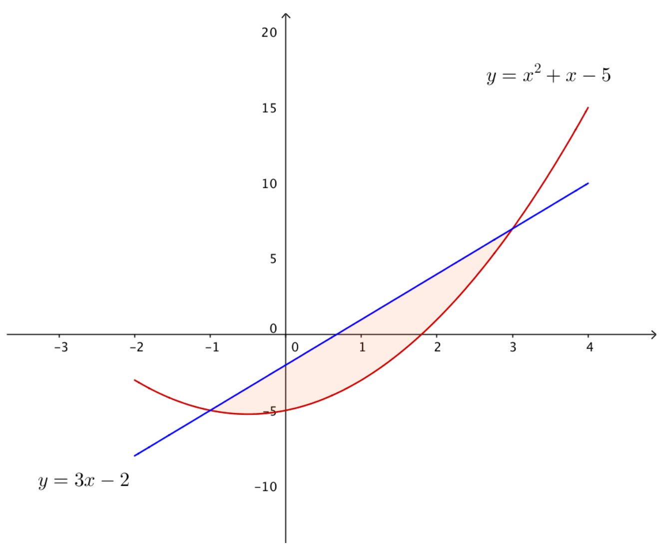
Example 4.2. Find the area of the region enclosed by the curves $y = x^2 + x - 5$ and $y = 3x - 2$ in the $x - y$ plane.



Solution. Let $x^2 + x - 5 = 3x - 2 \quad \Rightarrow \quad x = -1, 3.$

The area is

$$\begin{aligned}\int_{-1}^{3} \big((3x-2)-(x^2+x-5)\big)\,dx &= \int_{-1}^{3}(-x^2+2x+3)\,dx \\ &= \left(-\frac{1}{3}x^3+x^2+3x\right)\Big|_{-1}^{3} \\ &= -\frac{1}{3}(27)+9+9-\left(\frac{1}{3}+1-3\right) \\ &= 10\frac{2}{3}.\end{aligned}$$

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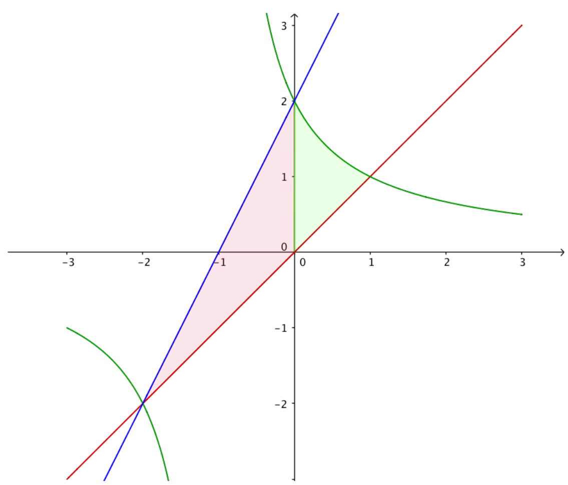
Example 4.3. Find the area bounded by the curves

$$y=f(x)=x, \quad y=g(x)=\frac{2}{x+1}, \quad \text{and} \quad y=h(x)=2x+2.$$



Solution. Area is

$$\begin{aligned}&\int_{-2}^{0}(h(x)-f(x))dx+\int_{0}^{1}(g(x)-f(x))dx \\ &= \int_{-2}^{0}(2x+2-x)+\int_{0}^{1}\left(\frac{2}{x+1}-x\right)dx \\ &= \left[\frac{x^2}{2}+2x\right]_{-2}^{0}+\left[2\ln|x+1|-\frac{x^2}{2}\right]_{0}^{1} \\ &= 2+(2\ln 2-\frac{1}{2})=\frac{3}{2}+\ln 4.\end{aligned}$$

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4.3 Other Applications

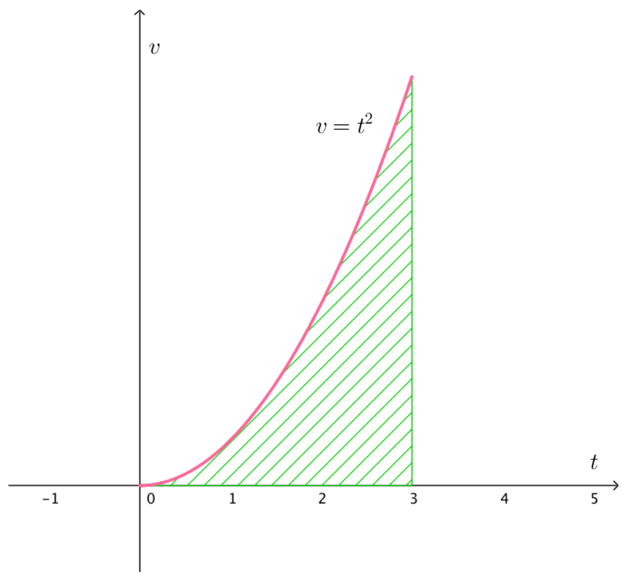
Example 4.4. An object moves along x-axis towards right with speed $v(t) = t^2$ m/s. Calculate the distance traveled from $t = 0$ to $t = 3$s.

Solution. Let $S(t)$ be the position at t. Then, $S'(t) = v(t) = t^2$.

The distance from $t = 0$ to $t = 3$ is

$$\underbrace{S(3) - S(0)}_{\text{total distance change}} = \int_0^3 \overbrace{S'(t)}^{\text{rate of change}} \, dt = \int_0^3 t^2 \, dt = \frac{1}{3}t^3 \Big|_0^3 = 9\text{m}$$

Geometrically,



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Example 4.5. Let $L(t)$ be the level of carbon monoxide (CO). Given that $L'(t) = 0.1t + 0.1$ parts per million (ppm). How much will the pollution change from $t = 0$ to $t = 3$?

Solution.

$$L(3) - L(0) = \int_0^3 L'(t)dt = \left[0.05t^2 + 0.1t\right]_0^1 = 0.75\text{ppm}.$$

■

Exercise 4.2. Let t be the time (in hour). Let $m(t)$ be the mass of a certain amount of protein. The protein is changed to an amino acid and cause a decrease in mass at a rate

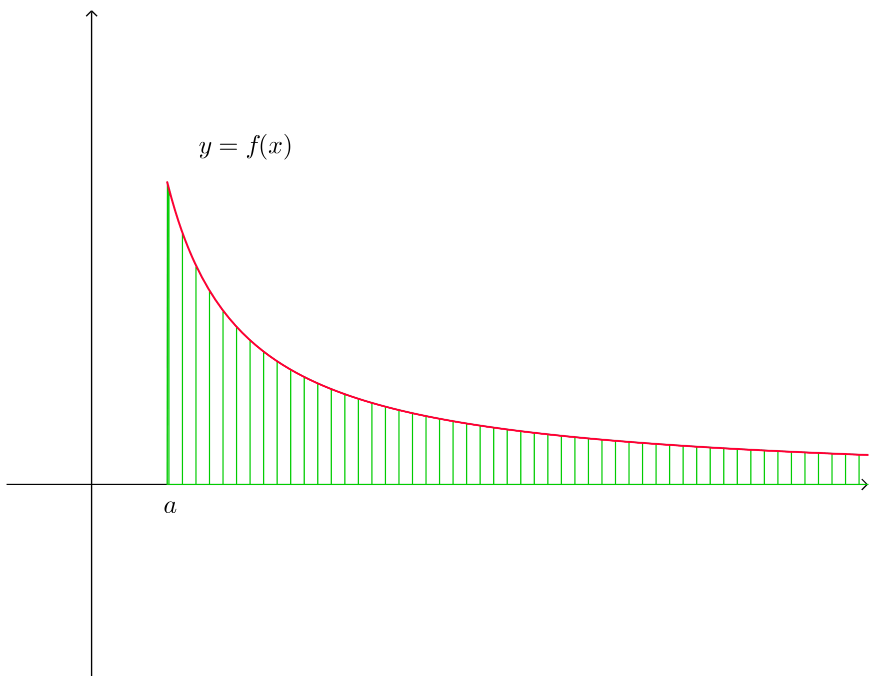
$$\frac{dm}{dt} = \frac{-2}{t+1} \text{g/hr}.$$

Find the decrease in mass of the protein from $t = 2$ to $t = 5$.

Ans: $-2 \ln 2$.

5 Improper Integrals

Question: How to find area of an unbounded region?



Definition 5.1. The following types of integrals are called "improper integrals" (of the first type). The integrals we have encountered previously, namely integrals of piecewise continuous functions over finite intervals, are "proper integrals".

Define

1.
$$\boxed{\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx}$$

if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

2.
$$\boxed{\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)dx}$$

if the limit exists, we say that the integral is **convergent**. Otherwise, **divergent**.

3. Let c be a fixed real number.

$$\boxed{\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx}$$

if both the two integrals on the right are convergent, we say that the integral is **convergent**. Otherwise, **divergent**.

Example 5.1.

1. $\int_{0}^{+\infty} e^{-x}dx = \lim_{b\to+\infty}\int_{0}^{b} e^{-x}dx = \lim_{b\to+\infty} -e^{-x}\Big|_0^b = \lim_{b\to+\infty}(e^0 - e^{-b}) = \lim_{b\to+\infty}(1-e^{-b}) = 1,$ convergent.
2. $\int_{1}^{+\infty} \frac{1}{x}dx = \lim_{b\to+\infty}\int_{1}^{b}\frac{1}{x}dx = \lim_{b\to+\infty}\ln x|_1^b = \lim_{b\to+\infty}\ln b - \ln 1 = \lim_{b\to+\infty}\ln b = +\infty,$ divergent.
3. $\int_{1}^{+\infty}\frac{1}{x^2}dx = \lim_{b\to+\infty}\int_{1}^{b}\frac{1}{x^2}dx = \lim_{b\to+\infty}\left(-\frac{1}{x}\right)\Big|_1^b = \lim_{b\to+\infty}\left(1-\frac{1}{b}\right) = 1,$ convergent.
4. $\int_{1}^{+\infty}\frac{1}{\sqrt{x}}dx = \lim_{b\to+\infty}\int_{1}^{b}\frac{1}{\sqrt{x}}dx = \lim_{b\to+\infty} 2\sqrt{x}\Big|_1^b = \lim_{b\to+\infty} 2(\sqrt{b}-1) = +\infty,$ divergent.
5. $\int_{-\infty}^{0} e^x dx$

Example 5.2. Compute $\int_0^{+\infty} \frac{dx}{(x+1)(3x+2)}$.

Solution.

$$\frac{1}{(x+1)(3x+2)} = \frac{3}{3x+2} - \frac{1}{x+1}.$$

Hence

$$\int_0^b \frac{dx}{(x+1)(3x+2)} = [\ln|3x+2| - \ln|x+1|]_0^b$$

$$= \ln|3b+2| - \ln|b+1| - \ln|2| = \ln\frac{|3b+2|}{|b+1|} - \ln 2.$$

Because

$$\lim_{b\to+\infty} \frac{|3b+2|}{|b+1|} = \lim_{b\to+\infty} \frac{|3b+2| \times \frac{1}{|b|}}{|b+1| \times \frac{1}{|b|}}.$$

$$\lim_{b\to+\infty} \frac{|3+\frac{2}{b}|}{|1+\frac{1}{b}|} = \frac{3}{1} = 3.$$

Therefore

$$\lim_{b\to+\infty} \int_0^b \frac{dx}{(x+1)(3x+2)} = \ln 3 - \ln 2.$$

■

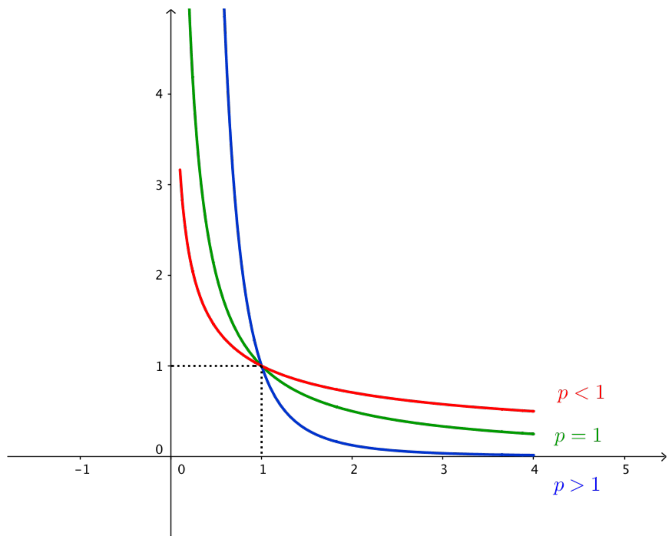
Exercise 5.1. Let $p > 1$. Prove that

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{p-1}, & \text{if} \quad p > 1, \qquad \text{convergent} \\ +\infty, & \text{if} \quad 0 < p \le 1, \qquad \text{divergent}. \end{cases}$$

Remark. From the above exercise,

1. $\lim_{x\to+\infty} f(x) = 0 \quad \not\Rightarrow \quad \int_1^{+\infty} f(x)\,dx$ is convergent.
2. For all $p > 0$, $\frac{1}{x^p} \to 0$ as $x \to +\infty$. However, only for $p > 1$, $\frac{1}{x^p}$ decays fast enough to guarantee the total area $\int_1^{+\infty} \frac{1}{x^p}\,dx$ is finite.

Remark. All the integration techniques can be applied, e.g. integration by substitution,...



Example 5.3. Compute $\int_{-\infty}^{1} xe^x\,dx$. (integration by parts)

Solution.

$$\int_{-\infty}^{1} xe^x\,dx = \lim_{a\to-\infty}\int_a^1 xe^x\,dx.$$

$$\int xe^x\,dx = \int x d(e^x) = xe^x - \int e^x\,dx = (x-1)e^x + C.$$

$$\begin{aligned}
\int_{-\infty}^{1} xe^x\,dx &= \lim_{a\to-\infty} (x-1)e^x|_a^1 \\
&= \lim_{a\to-\infty} (1-a)e^a \quad \infty\cdot 0 \quad \text{indeterminate form} \\
&= \lim_{a\to-\infty} \frac{1-a}{e^{-a}} \quad \frac{\infty}{\infty} \\
&= \lim_{a\to-\infty} \frac{-1}{-e^{-a}} \quad \text{L'Hôpital's rule} \\
&= 0.
\end{aligned}$$

■

Exercise 5.2. $\int_{-\infty}^{1} x^2e^x dx = e$

Example 5.4. Compute $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx$. (integration by substitution)

Solution. Using the substitution $u = 1 + x^2$, we have

$$\int \frac{x}{(1+x^2)^2}\,dx = \frac{-1}{2(1+x^2)} + C.$$

Thus

$$\int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2}$$

and

$$\int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = -\frac{1}{2}.$$

Hence

$$\int_{-\infty}^{+\infty} \frac{x}{(1+x^2)^2}\,dx = \int_0^{+\infty} \frac{x}{(1+x^2)^2}\,dx + \int_{-\infty}^{0} \frac{x}{(1+x^2)^2}\,dx = \frac{1}{2} + \left(-\frac{1}{2}\right) = 0.$$

■

Fact: If $0 \le f(x) \le g(x)$ on the interval of integration (a, b) (allowing a, b to be $\pm\infty$), then

- If $\displaystyle\int_a^b g(x)dx$ converges, then $\displaystyle\int_a^b f(x)dx$ converges.
- If $\displaystyle\int_a^b f(x)dx$ diverges, then $\displaystyle\int_a^b f(x)dx$ diverges.

Example 5.5. Determine whether $\displaystyle\int_0^{\infty} x^n e^{-x} dx$ is convergent.

Definition 5.2 (Improper integrals of Type 2)**.** The improper integrals defined in Definition 5.1 has infinite intervals of integration, but the values of the integrand are finite on the intervals of the integration. We also generalize definite integrals where the integrand may go to $\pm\infty$ over the interval of integration.

Suppose that $f(x)$ is continuous on (a, b), but $\lim_{x\to b^-} f(x) = \pm\infty$. Then we define:

$$\int_a^b f(x)dx := \lim_{c\to b^-} \int_a^c f(x)dx.$$

Similarly, if $\lim_{x\to a^+} f(x) = \pm\infty$,

$$\int_a^b f(x)dx := \lim_{c\to a^+} \int_c^b f(x)dx.$$

Example 5.6.

1. $\displaystyle\int_0^1 \frac{1}{x^p}dx$
2. $\displaystyle\int_0^1 \frac{1}{\ln x}dx$
3. (mixed type) $\displaystyle\int_{-\infty}^1 \frac{1}{x^3}dx$